Isoperimetric Inequalities and Sharp Estimate for Positive Solution of Sublinear Elliptic Equations [*]

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Abstract: In this paper, we prove some isoperimetric inequalities and give a sharp bound for the positive solution of sublinear elliptic equations.

Key words: Isoperimetric inequality, Schwarz symmetrization, positive solution, sublinear equation.

1 Introduction and Main Results

Let $\Omega \subset R^n$ be a bounded domain whose boundary $\partial\Omega$ is assumed to be of Lipschitz type. Assume that $0 < q < 1$. We consider the following problem.

$$\begin{cases} -\Delta u = u^q, & x \in \Omega, \\ u > 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

The purpose of this paper is to prove some isoperimetric inequalities and give sharp bound for the solution of problem (1.1) by making use of rearrangement method.

There are a lot of materials on isoperimetric inequalities for eigenvalues and eigenfunctions of elliptic operators. For the isoperimetric inequalities on eigenvalues of elliptic operators we refer to [4, 3, 11, 16, 7, 8, 13, 14, 21, 22, 26, 29] and on eigenfunctions we refer to [24, 25, 10, 9, 6, 19, 20]. The first result on isoperimetric inequality for eigenfunctions of Laplace operator was obtained by Payne and Rayner in [24] . In 1972, Payne and Rayner considered in [24] the following eigenvalue problem defined on bounded domains in R^2

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and prove that for the first eigenvalue $\lambda_1(\Omega)$ and the first eigenfunction $\varphi_1(x)$ of problem (1.2), the following inequality holds

$$\left(\int_\Omega |\varphi_1| dA\right)^2 \geq \frac{4\pi}{\lambda_1(\Omega)} \int_\Omega \varphi_1^2 dA, \tag{1.3}$$

with equality if and only if Ω is a disk.

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Unfortunately, the argument used by Payne and Rayner works only for the case $n = 2$. Kohnler-Jobin [19, 20] and G.Chiti [10] generalized the Payne and Rayner's inequality (1.3) to arbitrary dimension n by employing the Schwarz symmetrization method. It is by now well known that the Schwarz symmetrization method is very useful for the estimate of sharp bound of solutions to elliptic and parabolic equations, and has been extensively studied since the pioneer works of Weinberger [30], Talenti [27] and Bandle [5]. See for example [28, 23, 1, 2] for more details. The basic idea in the use of the symmetrization method is to compare the orignal problem with an auxiliary problem defined on a suitable ball. Let Ω^* be the Schwarz symmetrization of Ω, that is Ω^* is a ball in R^n with center at 0 and such that Ω^* and Ω have same volume. The auxiliary problem used by Kohnler-Jobin to generalize inequality (1.3) can be read as

$$\begin{cases} -\Delta\varphi = \alpha\varphi + 1 & \text{in } \Omega^*, \\ \varphi = 0 & \text{on } \partial\Omega^*, \end{cases} \tag{1.4}$$

with $-\infty < \alpha < \lambda_1(\Omega)$.

Whereas, G.Chiti used an auxiliary problem defined on a ball smaller than Ω^* which can be read as

$$\begin{cases} -\Delta z = \lambda z, & x \in B_r(0), \\ z = 0, & x \in \partial B_r(0), \end{cases} \tag{1.5}$$

where $r = \sqrt{\frac{\lambda_1(\Omega^*)}{\lambda_1(\Omega)}} R^*$ and R^* is the radius of Ω^*.

It follows from the famous Faber-Krahn inequality that $r \leq R^*$, and hence $B_r(0)$ is smaller than Ω^*. Furthermore, an easy computation implies that the first eigenvalue of problem (1.5) is $\lambda_1(\Omega)$.

Compared with the auxiliary problem used by Kohnler-Jobin, the problem used by G.Chiti is more natural and extendable for other situations.

Let $\varphi_1(x)$ be the first eigenfunction of problem (1.2), and $z_1(x)$ be the first eigenfunction of problem (1.5). If we normalize $\varphi_1(x)$ and $z_1(x)$ so that $\int_\Omega \varphi_1^p(x)dx = \int_{B_r(0)} z_1^p(x)dx$ for $p > 1$, then a celebrate result established by G.Chiti in [10] can be stated as

Conclusion A. There exists an unique point $s_0 \in (0,\ |B_r(0)|)$ such that

$$\begin{cases} z_1^*(s) > \varphi_1^*(s), & \text{for } s \in (0,\ s_0), \\ z_1^*(s) < \varphi_1^*(s), & \text{for } s \in (s_0,\ |B_r(0)|). \end{cases}$$

where $z_1^*(s)$ and $\varphi_1^*(s)$ are the decreasing rearrangement of $z_1(x)$ and $\varphi_1(x)$ respectively, and $|B_r(0)|$ denotes the volume of $B_r(0)$.

By making use of conclusion A, Chiti proved a reverse Holder inequality for the first eigenfunction of problem (1.2) which, in turn, is an isoperimetric inequality and more stronger than inequality (1.3). It is worth pointing out that a most important application of conclusion A can be found in the proof of the famous P.P.W conjecture (see [3]).

Contrast to the eigenvalue problem, there are few results on the isoperimetric inequalities for solutions of semilinear elliptic problem. This is the motivation of our study of the isoperimetric inequalities for the solution of problem (1.1). Our method is adapted from G. Chiti's paper [10] by carefully choosing the comparison problem.

To state our results, we introduce the following auxiliary problem

$$\begin{cases} -\Delta h = h^q, & x \in \Omega^*, \\ h > 0, & x \in \Omega^* \\ h = 0, & x \in \partial\Omega^*. \end{cases} \tag{1.6}$$

where Ω^* is the Schwarz symmetrization of Ω.

Let $\sigma_1 = \frac{2(1+q)k+(1-q^2)n}{n+2-(n-2)q}$ and $\sigma_2 = \frac{2(1+q)}{n+2-(n-2)q}$ be fixed. Then our main result can be stated as

Theorem 1.1. Let $u(x)$ be the unique solution of problem (1.1) and $h(x)$ be the unique solution of problem (1.6). Then for any $k \geq q+1$, we have

$$\int_\Omega u^k(x)dx \leq C(q,k,\Omega^*)\|u\|^{\sigma_1}_{L^{q+1}(\Omega)}. \tag{1.7}$$

Consequentely

$$\max_{x\in\Omega} u(x) \leq C(q,\Omega^*)\|u\|^{\sigma_2}_{L^{q+1}(\Omega)}, \tag{1.8}$$

where $C(q,k,\Omega^*) = \int_{\Omega^*} h^k(x)dx/\|h\|^{\sigma_1}_{L^{q+1}(\Omega^*)}$ and $C(q,\Omega^*) = \max_{x\in\Omega^*} h(x)/\|h\|^{\sigma_2}_{L^{q+1}(\Omega^*)}$. Moreover, the equality holds in each of inequalities (1.7) and (1.8) if and only if Ω is a ball.

By Theorem 1.1 and a Faber-Krahn type inequality proved in section 3 Lemma 3.2, we have

Corollary 1.2. Let $u(x)$ be the unique solution of problem (1.1) and $h(x)$ be the unique solution of problem (1.6). Then for any $k \geq q+1$, we have

$$\int_\Omega u^k(x)dx \leq \int_{\Omega^*} h^k(x)dx, \tag{1.9}$$

and

$$\max_{x\in\Omega} u(x) \leq \max_{x\in\Omega^*} h(x). \tag{1.10}$$

Moreover, the equality holds in each of inequalities (1.9) and (1.10) if and only if Ω is a ball.

Thanks to Corollary 1.2 and an explicit bound of solution of problem (1.6), we have

Corollary 1.3. Let $u(x)$ be the unique solution of problem (1.1), and ω_n is the volume of unit ball in R^n. Then

$$\max_{x\in\Omega} u(x) \leq \left[\frac{|\Omega|}{\omega_n(2n)^{\frac{n}{2}}}\right]^{\frac{2}{(1-q)n}} \tag{1.11}$$

with equality only if Ω is a ball.

Remark 1.4. Let $u(x)$ be the unique solution of problem (1.1). If $|\Omega| < \omega_n(2n)^{\frac{n}{2}}$, then it follows from Corollary 1.3 that $u(x) \to 0$ uniformly on Ω when $q \to 1^-$. It is interest to know the asymptotic behavior of $u(x)$ when $|\Omega| \geq \omega_n(2n)^{\frac{n}{2}}$ and $q \to 1^-$. It is also interest to know the asymptotic behavior of $u(x)$ when $q \to 0^+$.

Remark 1.5. All results of this paper can be generalized to p-Laplace equation with some modification of our method (see [12]).

The paper is organized as follows: As preliminary, we give some basic facts about the rearrangement of functions in section 2. In section 3, we prove a Chiti type comparison result which is essential to the proof of our main results. The proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3 are given in section 4.

2 Preliminary

In this section, we recall some basic facts about the rearrangement of functions and the existence and uniqueness result of problem (1.1).

Let Ω be a bounded domain in R^n. The Schwartz symmetrization Ω^* of Ω is a ball in R^n with radius R^* and centered at 0 such that $|\Omega^*| = |\Omega|$. Here, $|\Omega|$ denotes the Lebesgue measure of Ω. If we denote by ω_n the volume of unit ball in R^n, then it is easy to see

$$R^* = \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}}.$$

Let $f:\ \Omega \mapsto R$ be a nonnegative measurable function. For any $t \geq 0$. The level set Ω_t of f at the level t is defined by

$$\Omega_t \doteq \{x \in \Omega :\ f(x) > t\}, \qquad t \geq 0.$$

The distribution function of f is given by

$$\mu_f(t) = |\Omega_t| = \text{meas}\{x \in \Omega :\ f(x) > t\}, \qquad t \geq 0.$$

Obviously, $\mu_f(t)$ is a monotonically decreasing function of t and $\mu_f(t) = 0$ for $t \geq \text{ess.}\sup.f$, while $\mu_f(t) = |\Omega|$ for $t = 0$.

Definition 2.1. Let Ω be a bounded domain in R^n, $f:\ \Omega \mapsto R$ be a nonnegative measurable function. Then the decreasing rearrangement f^* of f is a function defined on $[0,\ \infty)$ by

$$f^*(s) = \begin{cases} \text{ess.}\sup.f & \text{for } s = 0 \\ \inf\{t > 0 | \mu_f(t) < s\} & \text{for } s > 0. \end{cases}$$

Obviously, $f^*(s) = 0$ for $s \geq |\Omega|$. The increasing rearrangement f_* of f is defined by $f_*(s) = f^*(|\Omega| - s)$ for $s \in (0,\ +\infty)$.

Definition 2.2. Let Ω be a bounded domain in R^n, $f:\ \Omega \mapsto R$ be a nonnegative measurable function. Then the decreasing Schwarz symmetrization $f^\star$ of f is a function defined by

$$f^\star(x) = f^*(\omega_n |x|^n) \quad \text{for } x \in \Omega^*.$$

There are many fine properties of rearrangement. Here, we only collect some important properties needed in this paper.

Proposition 2.3. Let $f:\ \Omega \mapsto R$ be a nonnegative measurable function. Then, $f,\ f^*$ and $f^\star$ are all equimeasurable and

$$\int_\Omega f dx = \int_0^{|\Omega|} f^*(s) ds = \int_{\Omega^*} f^\star(x) dx.$$

Moreover, for any Borel measurable function $F:\ R \mapsto R$, there holds

$$\int_\Omega F(f(x)) dx = \int_0^{|\Omega|} F(f^*(s)) ds = \int_{\Omega^*} F(f^\star(x)) dx.$$

Proposition 2.4. If $f: [0, l] \mapsto R$ is nonnegative and non-increasing, then $f = f^*$ *a.e.*

Proposition 2.5. If $\psi: R \mapsto R$ is a non-decreasing function, then

$$\psi(f^*) = (\psi(f))^*, \quad \psi(f^\star) = (\psi(f))^\star$$

for any nonnegative measurable function $f: \Omega \mapsto R$.

Proposition 2.6. Let $f \in L^p(\Omega),\ g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_0^{|\Omega|} f^*(s) g_*(s) ds \le \int_\Omega f(x) g(x) dx \le \int_0^{|\Omega|} f^*(s) g^*(s) ds,$$

$$\int_{\Omega^*} f^\star(x) g_\star(x) dx \le \int_\Omega f(x) g(x) dx \le \int_{\Omega^*} f^\star(x) g^\star(x) dx.$$

Consequently

$$\int_E f(x) dx \le \int_0^{|E|} f^*(s) ds = \int_{E^*} f^\star(x) dx.$$

for any measurable set $E \subset \Omega$.

Proposition 2.7. If $f \in H_0^1(\Omega)$, then $f^\star \in H_0^1(\Omega^*)$ and

$$\int_\Omega |\nabla f|^2 dx \ge \int_{B_{R^*}(0)} |\nabla f^\star|^2 dx$$

where $B_{R^*}(0) = \Omega^*$. Moreover, the equality holds if and only if Ω is a ball.

The proof of all propositions mentioned above can be found in [18, 17].

Proposition 2.8 ([15]). Let $M,\ \alpha,\ \beta$ be real numbers such that $0 < \alpha \le \beta$ and $M > 0$. Let $f,\ g$ be real functions in $L^\beta([0,\ M])$. If the decreasing rearrangements of f and g satisfy the inequality

$$\int_0^s f^{*^\alpha}(t) dt \le \int_0^s g^{*^\alpha}(t) dt \quad \text{for} \quad s \in [0,\ M],$$

then

$$\int_0^M f^{*^\beta}(t) dt \le \int_0^M g^{*^\beta}(t) dt.$$

The following result may be well known. However, for the reader's convenience, we give a proof here.

Proposition 2.9. Problem (1.1) has an unique solution.

Proof. Let $h(x)$ be the unique solution of

$$\begin{cases} -\Delta h = 1, & x \in \Omega, \\ h = 0, & x \in \partial\Omega. \end{cases}$$

Choose M_0 so that $M_0 > M_0^q \max_{x \in \Omega} h^q(x)$, this is possible since $0 < q < 1$.

Let $v_0(x) = M_0 h(x)$, then

$$-\Delta v_0 = -M_0 \Delta h(x) = M_0 > M_0^q \max_{x \in \Omega} h^q(x) \ge v_0^q.$$

This implies that $v_0(x)$ is sup-solution of problem (1.1).

Let $\varphi_1(x)$ be the first eigenfunction of the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega. \end{cases}$$

We choose $\varphi_1(x)$ so that

$$\varphi_1(x) > 0, \quad \max_{x\in\Omega} \varphi_1(x) = 1.$$

Let $v_{\eta_0} = \eta_0\varphi_1(x)$, then

$$-\Delta v_{\eta_0} = -\eta_0\Delta\varphi_1(x) = \lambda_1\eta_0\varphi_1(x).$$

Since $0 < q < 1$, we can choose η_0 small enough such that

$$\lambda_1\eta_0\varphi_1(x) \le \eta_0^q\varphi_1^q(x) = v_{\eta_0}^q.$$

Hence v_{η_0} is a sub-solution of problem (1.1). Choosing η_0 even more smaller, we can assume that $v_{\eta_0} \le v_0(x)$. Then by the sub- and super- solution method, we know that problem (1.1) has at least one solution $u(x)$ which satisfies $v_{\eta_0} \le u(x) \le v_0(x)$.

To prove the uniqueness, we assume that $u_1(x)$ and $u_2(x)$ are any two solutions of problem (1.1). It is obvious that for $b > 0$ small enough, we have

$$u_1(x) > bu_2(x), \qquad x \in \Omega.$$

Let

$$b_0 = \sup\{b|\ u_1(x) > bu_2(x), \quad x \in \Omega\}.$$

Then

$$u_1(x) \ge b_0u_2(x), \qquad x \in \Omega.$$

and there exists at least one point $x_0 \in \Omega$ such that

$$u_1(x_0) = b_0u_2(x_0), \qquad x \in \Omega. \tag{2.1}$$

If $b_0 < 1$, then $v_0 = b_0u_2(x)$ satisfies

$$-\Delta v_0 = -b_0\Delta u_2(x) = b_0u_2^q(x) < b_0^qu_2^q(x) = v_0^q.$$

Let $w = u_1(x) - v_0(x)$, then $w(x)$ satisfies

$$\begin{cases} -\Delta w > u_1^q(x) - v_0^q(x) \ge 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases}$$

It follows from the strong maximum principle that $w(x) > 0,\ x \in \Omega$. Hence

$$u_1(x) > b_0u_2(x), \qquad x \in \Omega.$$

This contradicts (2.1). Thus we must have $b_0 \ge 1$ and

$$u_1(x) \ge u_2(x), \quad \text{for } x \in \Omega.$$

Changing the position of $u_1(x)$ and $u_2(x)$, a similar argument implies that

$$u_2(x) \ge u_1(x), \quad \text{for } x \in \Omega.$$

Consequently,

$$u_1(x) \equiv u_2(x), \quad \text{for } x \in \Omega.$$

This means that problem (1.1) has only one solution.

3 Chiti Type Comparison Result

Let Ω be a bounded domain in R^n, and $\|\cdot\|_{L^{q+1}(\Omega)}$ denote the norm of space $L^{q+1}(\Omega)$. We define

$$S_q(\Omega) = \inf_{v \in H_0^1(\Omega)} \{ \int_\Omega |\nabla v|^2 dx \Big| \ \|v\|^2_{L^{q+1}(\Omega)} = 1. \}$$

It is easy to prove that $S_q(\Omega)$ can be achieved by an unique positive function $v(x)$. Moreover, $v(x)$ satisfies

$$\begin{cases} -\Delta v(x) = S_q(\Omega) v^q(x), & x \in \Omega, \\ v(x) > 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \\ \int_\Omega v^{q+1}(x) dx = 1. \end{cases} \tag{3.1}$$

In this section, we prove a Chiti type comparison result for problem (3.1). To this end, we need some lemmas first.

Lemma 3.1. For any $\lambda > 0$ and $\lambda \neq S_q(\Omega)$, the following problem has no solution

$$\begin{cases} -\Delta f(x) = \lambda f^q(x), & x \in \Omega, \\ f(x) > 0, & x \in \Omega, \\ f(x) = 0, & x \in \partial\Omega, \\ \int_\Omega f^{q+1}(x) dx = 1. \end{cases} \tag{3.2}$$

Proof. We prove Lemma 3.1 by contradiction. Assume that problem (3.2) has a solution f_{λ_0} for some $\lambda_0 > 0$ and $\lambda_0 \neq S_q(\Omega)$. Then, it is easy to check that $\widetilde{f} = \lambda_0^{\frac{1}{q-1}} f_{\lambda_0}$ is a solution of problem (1.1) which satisfies

$$\int_\Omega \widetilde{f}^{q+1}(x) dx = \lambda_0^{\frac{q+1}{q-1}}.$$

On the other hand, if we denote by $v(x)$ the minimizer of $S_q(\Omega)$, then $\widetilde{v} = S_q^{\frac{1}{q-1}}(\Omega) v(x)$ is also a solution of problem (1.1) which satisfies

$$\int_\Omega \widetilde{v}^{q+1}(x) dx = S_q^{\frac{q+1}{q-1}}(\Omega).$$

It is obvious that $\widetilde{v} \neq \widetilde{f}$ due to $\lambda_0 \neq S_q(\Omega)$. Hence problem (1.1) has at least two solutions $\widetilde{v}$ and $\widetilde{f}$. This contradicts Proposition 2.9.

Lemma 3.2. $S_q(\Omega) \geq S_q(\Omega^*)$ with equality if and only if Ω is a ball.

Proof. Let $v(x)$ be the minimizer of $S_q(\Omega)$ and $v^\star(x)$ be its Schwartz symmetrization. Then by Proposition 2.3 and Proposition 2.7, we have

$$\int_\Omega |\nabla v|^2 dx \geq \int_{\Omega^*} |\nabla v^\star|^2 dx,$$

$$\int_\Omega v^{q+1}(x) dx = \int_{\Omega^*} (v^\star)^{q+1}(x) dx = 1.$$

Hence, by the definition of $S_q(\Omega^*)$, we have

$$S_q(\Omega^*) \le \int_{\Omega^*} |\nabla v^\star|^2 dx \le \int_\Omega |\nabla v|^2 dx = S_q(\Omega).$$

If $S_q(\Omega^*) = S_q(\Omega)$, then $\int_{\Omega^*} |\nabla v^\star|^2 dx = \int_\Omega |\nabla v|^2 dx$. Hence, by Proposition 2.7, we know that Ω is a ball.

Let $\sigma_3 = \frac{q+1}{n+2-(n-2)q}$. Then the following lemma holds

Lemma 3.3. Let $v(x)$ be the minimizer of $S_q(\Omega^*)$ and $r_* = \left(\frac{S_q(\Omega^*)}{S_q(\Omega)}\right)^{\sigma_3} R^*$. Then $S_q(B_{r_*}(0)) = S_q(\Omega)$ and the minimizer of $S_q(B_{r_*}(0))$ is $z(y) = \left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} v(\frac{R^*}{r_*}y)$ for $y \in B_{r_*}(0)$.

Proof. Since $v(x)$ is the minimizer of $S_q(\Omega^*)$, $v(x)$ satisfies

$$\begin{cases} -\Delta v(x) = S_q(\Omega^*) v^q(x), & x \in \Omega^*, \\ v(x) > 0, & x \in \Omega^*, \\ v(x) = 0, & x \in \partial\Omega^*, \\ \int_{\Omega^*} v^{q+1}(x) dx = 1. & \end{cases}$$

Let $x = \frac{R^*}{r_*} y$ and $H(y) = v(\frac{R^*}{r_*}y)$. Then

$$\frac{\partial H}{\partial y_i} = \frac{R^*}{r_*}\frac{\partial v}{\partial x_i},$$
$$\frac{\partial^2 H}{\partial y_i^2} = (\frac{R^*}{r_*})^2 \frac{\partial^2 v}{\partial x_i^2}.$$

Hence

$$-\Delta H(y) = -(\frac{R^*}{r_*})^2 \Delta v = (\frac{R^*}{r_*})^2 S_q(\Omega^*) H^q(y), \quad y \in B_{r_*}(0).$$

Noting that

$$\begin{aligned} 1 = \int_{\Omega^*} v^{q+1}(x) dx &= (\frac{R^*}{r_*})^n \int_{B_{r_*}(0)} H^{q+1}(y) dy \\ &= \int_{B_{r_*}(0)} \left[(\frac{R^*}{r_*})^{\frac{n}{q+1}} H(y)\right]^{q+1} dy, \end{aligned}$$

if we let $z(y) = (\frac{R^*}{r_*})^{\frac{n}{q+1}} H(y) = (\frac{R^*}{r_*})^{\frac{n}{q+1}} v(\frac{R^*}{r_*}y)$, then $z(y)$ satisfies

$$\begin{cases} -\Delta z(y) = \left(\frac{R^*}{r_*}\right)^{\frac{1}{\sigma_3}} S_q(\Omega^*) z^q(y), & y \in B_{r_*}(0), \\ z(y) > 0, & y \in B_{r_*}(0), \\ z(y) = 0, & y \in \partial B_{r_*}(0), \\ \int_{B_{r_*}(0)} z^{q+1}(y) dy = 1. & \end{cases}$$

Hence, by Lemma 3.1, we have

$$S_q(B_{r_*}(0)) = \left(\frac{R^*}{r_*}\right)^{\frac{1}{\sigma_3}} S_q(\Omega^*) = S_q(\Omega).$$

and the minimizer of $S_q(B_{r_*}(0))$ is $z(y) = \left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} v(\frac{R^*}{r_*}y)$. This completes the proof of Lemma 3.3.

By Lemma 3.2 and the definition of r_*, we have $B_{r_*}(0) \subset \Omega^*$ with equality if and only if Ω is a ball. Let $M = |\Omega|$ and $M_* = |B_{r_*}(0)|$, then $M_* \leq M$. The main result of this section is the following Chiti type comparison result.

Theorem 3.4. Let $v(x)$ be the minimizer of $S_q(\Omega)$ and $z(x)$ be the minimizer of $S_q(B_{r_*}(0))$. If we denote by $v^*(s)$ the decreasing rearrangement of $v(x)$, and $z^*(s)$ the decreasing rearrangement of $z(x)$, then there exists an unique point $s_0 \in (0,\ M_*)$ such that

$$\begin{cases} z^*(s) > u^*(s) & \text{for } s \in [0,\ s_0) \\ \\ z^*(s) < u^*(s) & \text{for } s \in (s_0,\ M_*]. \end{cases}$$

Proof. Since $u(x)$ is the minimizer of $S_q(\Omega)$, it is easy to see that $u(x)$ satisfies

$$\begin{cases} -\Delta u(x) = S_q(\Omega) u^q(y), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

From this, we can prove that the decreasing rearrangement $u^*(s)$ of $u(x)$ satisfies

$$-\frac{du^*(s)}{ds} \leq S_q(\Omega) n^{-2} \omega_n^{\frac{-2}{n}} s^{-\frac{2(n-1)}{n}} \int_0^s (u^*)^q(t)dt \qquad a.e. \text{ in } [0,\ M], \tag{3.4}$$

In fact, integrating the first equation in (3.3) over $\Omega_t = \{x \in \Omega \mid u(x) > t\}$, we have

$$-\int_{\partial\Omega_t} \frac{\partial u(x)}{\partial \nu} ds = S_q(\Omega) \int_{\Omega_t} u^q dx. \tag{3.5}$$

Since $\partial\Omega_t = \{x \in$
$Omega \mid u(x) = t\}$, we have

$$-\int_{\partial\Omega_t} \frac{\partial u(x)}{\partial \nu} ds = \int_{\partial\Omega_t} |\nabla u| ds. \tag{3.6}$$

Noting that

$$\int_{\partial\Omega_t} |\nabla u| ds \int_{\partial\Omega_t} \frac{ds}{|\nabla u|} \geq |\partial\Omega_t|^2.$$

It follows from the isoperimetric inequality

$$\int_{\partial\Omega_t} |\nabla u| ds \int_{\partial\Omega_t} \frac{ds}{|\nabla u|} \geq n^2 \omega_n^{\frac{2}{n}} |\Omega_t|^{\frac{2(n-1)}{n}}. \tag{3.7}$$

By Co-area formula, we have

$$\mu(t) = |\Omega_t| = \int_{\Omega_t} dx = \int_t^{+\infty} \int_{\partial\Omega_t} \frac{ds}{|\nabla u|}.$$

Consequently,

$$\frac{d\mu(t)}{dt} = -\int_{\partial\Omega_t} \frac{ds}{|\nabla u|}. \tag{3.8}$$

From (3.5), (3.6), (3.7) and (3.8), we obtain

$$\frac{n^2\omega_n^{\frac{2}{n}}(\mu(t))^{\frac{2(n-1)}{n}}}{-\mu'(t)} \leq S_q(\Omega)\int_{\Omega_t} u^q dx. \tag{3.9}$$

Since $\Omega_t \subset \Omega$, we have

$$\int_{\Omega_t} u^q dx \leq \int_0^{|\Omega_t|} (u^q)^*(\tau)d\tau = \int_0^{\mu(t)} (u^*(\tau))^q d\tau. \tag{3.10}$$

Combing (3.9) with (3.10), we obtain

$$-\frac{1}{\mu'(t)} \leq S_q(\Omega)n^{-2}\omega_n^{\frac{-2}{n}}(\mu(t))^{-\frac{2(n-1)}{n}}\int_0^{\mu(t)}(u^*(\tau))^q d\tau.$$

Noticing that $u^*(s)$ is essentially an inverse of $\mu(t)$, we have

$$-\frac{du^*(s)}{ds} \leq S_q(\Omega)n^{-2}\omega_n^{\frac{-2}{n}}s^{-\frac{2(n-1)}{n}}\int_0^s (u^*)^q(\tau))d\tau.$$

This is just the desired conclusion of (3.4).

Since $S_q(B_{r_*}(0)) = S_q(\Omega)$, the minimizer $z(x)$ of $S_q(B_{r_*}(0))$ satisfies

$$\begin{cases} -\Delta z(x) = S_q(\Omega)z^q(x), & x \in B_{r_*}(0), \\ z(x) > 0, & x \in B_{r_*}(0), \\ z(x) = 0, & x \in \partial B_{r_*}(0). \end{cases} \tag{3.11}$$

Noticing that uniqueness result valid for (3.11), it is trivial to see that z is radial symmetry. That is $z(x) = z(|x|)$. Moreover, as a function of $s = \omega_n|x|^n$, $z(s)$ is decreasing. Hence, by making use of (3.11), Proposition 2.4 and Proposition 2.5, a similar argument to that used to derive (3.4) implies that

$$-\frac{dz^*(s)}{ds} = S_q(\Omega)n^{-2}\omega_n^{\frac{-2}{n}}s^{-\frac{2(n-1)}{n}}\int_0^s (z^*)^q(t)dt \qquad a.e. \text{ in } [0,\ M_*], \tag{3.12}$$

Now, Theorem 3.3 can be proved by making use of (3.4) and (3.12). To this end, we first note that there exists at least one point $s_0 \in (0,\ M_*)$ such that $u^*(s_0) = z^*(s_0)$ because of

$$\int_\Omega u^{q+1}(x)dx = \int_0^{M_*}(u^*)^{q+1}(s)ds = 1 = \int_0^{M_*}(z^*(s))^{q+1}ds = \int_{B_{r_*}(0)} z^{q+1}(x)dx.$$

Next, we prove that there exists only one point $s_0 \in (0,\ M_*)$ such that $u^*(s_0) = z^*(s_0)$. Otherwise, there would exist at least two points $s_1,\ s_2 \in (0,\ M_*)$ such that

$$u^*(s_1) = z^*(s_1), \quad u^*(s_2) = z^*(s_2).$$

This would imply that there exists an interval $[s_1,\ s_2] \subset [0,\ M_*)$ such that

$$\begin{cases} u^*(s_i) = z^*(s_i), & i = 1,\ 2; \\ u^*(s) > z^*(s), & s \in (s_1,\ s_2). \end{cases}$$

Let

$$w(s) = \begin{cases} z^*(s), & \text{if } \int_0^s (u^*(\tau))^q d\tau \leq \int_0^s (z^*(\tau))^q d\tau,\ s \in [0,\ s_1]; \\ u^*(s), & \text{if } \int_0^s (u^*(\tau))^q d\tau \geq \int_0^s (z^*(\tau))^q d\tau,\ s \in [0,\ s_1]; \\ u^*(s), & s \in [s_1,\ s_2]; \\ z^*(s), & s \in [s_2,\ M_*]. \end{cases}$$

Then, it is easy to verify that $w(s)$ satisfies

$$\begin{cases} -\frac{dw(s)}{ds} \leq S_q(\Omega) n^{-2}\omega_n^{-\frac{2}{n}} s^{-\frac{2(n-1)}{n}} \int_0^s w^q(t)dt, & a.e. \text{ in } [0,\ M_*], \\ w(s) > 0, & s \in (0,\ M_*), \\ w(M_*) = 0, \\ \|w\|_{L^{q+1}(0,\ M_*)} \geq 1. \end{cases} \tag{3.13}$$

Define

$$\eta(x) = \frac{w(\omega_n|x|^n)}{\|w(\omega_n|x|^n)\|_{q+1(B_{r_*}(0))}}.$$

Then, $\eta(x) \in W_0^{1,\ 2}(B_{r_*}(0))$ and $\|\eta(x)\|_{q+1(B_{r_*}(0))} = 1$. Since $\eta(x)$ is obviously not the minimizer of $S_q(B_{r_*}(0))$, we have

$$S_q(\Omega) = S_q(B_{r_*}(0)) < \int_{B_{r_*}(0)} |\nabla \eta(x)|^2 dx.$$

Since

$$\begin{aligned} \int_{B_{r_*}(0)} |\nabla \eta(x)|^2 dx &= n^2 \omega_n^{\frac{2}{n}} \int_0^{M_*} |\eta'(s)|^2 s^{\frac{2(n-1)}{n}} ds \\ &= \frac{n^2 \omega_n^{\frac{2}{n}}}{\|w\|^2_{q+1(B_{r_*}(0))}} \int_0^{M_*} |w'(s)|^2 s^{\frac{2(n-1)}{n}} ds, \end{aligned}$$

and

$$\begin{aligned} n^2 \omega_n^{\frac{2}{n}} \int_0^{M_*} |w'(s)|^2 s^{\frac{2(n-1)}{n}} ds &= n^2 \omega_n^{\frac{2}{n}} \int_0^{M_*} (-w'(s))(-w'(s)) s^{\frac{2(n-1)}{n}} ds \\ &\leq S_q(\Omega) \int_0^{M_*} (-w'(s)) \int_0^s w^q(\tau) d\tau ds \\ &= S_q(\Omega) \int_0^{M_*} w^{q+1}(s) ds \\ &= S_q(B_{r_*}(0)) \|w\|^{q+1}_{q+1(B_{r_*}(0))} \end{aligned}$$

We have

$$\int_{B_{r_*}(0)} |\nabla \eta(x)|^2 dx \leq S_q(B_{r_*}(0)) \|w\|^{q+1-2}_{q+1(B_{r_*}(0))} = S_q(B_{r_*}(0)) \|w\|^{q-1}_{q+1(B_{r_*}(0))}.$$

Thus

$$S_q(B_{r_*}(0)) < \int_{B_{r_*}(0)} |\nabla \eta(x)|^2 dx \leq S_q(B_{r_*}(0)) \|w\|^{q-1}_{q+1(B_{r_*}(0))}.$$

Noticing that $\|w\|_{q+1(B_{r_*}(0))} \geq 1$ and $q - 1 < 0$, we obtain

$$S_q(B_{r_*}(0)) < S_q(B_{r_*}(0)).$$

This is a contradiction.

Hence, there exists only one point $s_0 \in (0,\ M_*)$ such that $z^*(s_0) = u^*(s_0)$ and this implies that

$$\begin{cases} z^*(s) > u^*(s), & \text{for } s \in (0,\ s_0)), \\ z^*(s) < u^*(s), & \text{for } s \in (s_0,\ M_*). \end{cases}$$

So, we complete the proof of Theorem 3.3.

Corollary 3.4. Let $u(x)$ be the minimizer of $S_q(\Omega)$ and $z(x)$ be the minimizer of $S_q(B_{r_*}(0))$. Then for any $k \geq q+1$, there holds

$$\int_\Omega u^k dx \leq \int_{B_{r_*}(0)} z^k(x) dx.$$

It follows that

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in B_{r_*}(0)} z(x).$$

Moreover, the equality holds in the above two inequalities if and only if Ω is a ball.

Proof. By the proposition of rearrangement, we have

$$\int_0^M (u^*)^{q+1}(s)ds = 1 = \int_0^{M_*} (z^*(s))^{q+1} ds.$$

Hence

$$\int_0^{M_*} (u^*)^{q+1}(s)ds \leq \int_0^{M_*} (z^*(s))^{q+1} ds.$$

Let s_0 be the point in $(0,\ M_*)$ determined in Theorem 3.3. Then

$$\int_{s_0}^{M_*} (u^*)^{q+1}(s)ds - \int_{s_0}^{M_*} (z^*)^{q+1}(s)ds \leq \int_0^{s_0} \left((z^*)^{q+1} - (u^*)^{q+1}\right)(s)ds.$$

Since $u^*(s) \geq z^*(s)$ for any $s \in [s_0,\ M_*]$. It follows that for any $s \in [s_0,\ M_*]$, there holds

$$\int_{s_0}^{s} \left((u^*)^{q+1} - (z^*)^{q+1}\right)(s)ds \leq \int_0^{s_0} \left((z^*)^{q+1} - (u^*)^{q+1}\right)(s)ds$$

Consequently,

$$\int_0^s (u^*)^{q+1}(\tau)d\tau \leq \int_0^s (z^*)^{q+1}(\tau)d\tau \quad \text{for any } s \in (0,\ M_*).$$

By the definition of $z^*(s)$, we have $z^*(s) = 0$ for $s \geq M_*$). Hence

$$\int_0^s (u^*)^{q+1}(\tau)d\tau \leq \int_0^s (z^*)^{q+1}(\tau)d\tau \quad \text{for any } s \in (0,\ M).$$

From this and Proposition 2.8, we have

$$\int_0^M (u^*)^k(s)ds \leq \int_0^{M_*} (z^*)^k(s)ds.$$

Noticing that

$$\int_\Omega u^k(x)dx = \int_0^M (u^*)^k(s)ds,$$

$$\int_{B_{r_*}(0)} z^k(x)dx = \int_0^{M_*} (z^*)^k(s)ds.$$

We obtain

$$\int_\Omega u^k(x)dx \leq \int_{B_{r_*}(0))} z^k(x)dx$$

for any $k \geq q+1$. This completes the proof of Corollary 3.4.

4 Proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3

In this section, we prove Theorem 1.1, Corollary 1.2 and Corollary 1.3. For simplicity, we always use the notations σ_1, σ_2 and σ_3 introduced in section 1 and section 3 in this section.

Proof of Theorem 1.1. Let $u(x)$ be the solution of problem (1.1). Then $v(x) = \frac{u(x)}{\|u\|_{L^{q+1}(\Omega)}}$ satisfies

$$\begin{cases} -\Delta v(x) = \|u\|_{L^{q+1}(\Omega)}^{q-1} v^q(x), & x \in \Omega, \\ v(x) > 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \\ \int_\Omega v^{q+1}(x)dx = 1. \end{cases}$$

Hence, by Lemma 3.1, we have $S_q(\Omega) = \|u\|_{L^{q+1}(\Omega)}^{q-1}$ and the minimizer of $S_q(\Omega)$ is $v(x)$.

Similarly, if $h(x)$ is the unique solution of problem (1.5). Then $S_q(\Omega^*) = \|h\|_{L^{q+1}(\Omega^*)}^{q-1}$ and the minimizer of $S_q(\Omega^*)$ is $\frac{h(x)}{\|h\|_{L^{q+1}(\Omega^*)}}$.

By the definition of r_*, we have $r_* = \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{(q-1)\sigma_3} R^*$. Moreover, by Lemma 3.3, we know that the minimizer of $S_q(B_{r_*}(0))$ is

$$z(x) = \left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} \frac{h(\frac{R^*}{r_*}x)}{\|h\|_{L^{q+1}(\Omega^*)}}.$$

Applying Corollary 3.4 to $v(x)$ and $z(x)$, we have, for any $k \geq q+1$, that

$$\begin{aligned} \int_\Omega u^k(x)dx &\leq \frac{\|u\|_{L^{q+1}(\Omega)}^k}{\|h\|_{L^{q+1}(\Omega^*)}^k} \int_{B_{r_*}(0)} \left(\frac{R^*}{r_*}\right)^{\frac{nk}{q+1}} h^k(\frac{R^*}{r_*}x)dx \\ &= \frac{\|u\|_{L^{q+1}(\Omega)}^k}{\|h\|_{L^{q+1}(\Omega^*)}^k} \left(\frac{R^*}{r_*}\right)^{\frac{nk}{q+1}-n} \int_{\Omega^*} h^k(x)dx \end{aligned}$$

Since

$$\left(\frac{R^*}{r_*}\right)^{\frac{nk}{q+1}-n} = \left(\frac{R^*}{r_*}\right)^{\frac{(k-q-1)n}{q+1}} = \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{-\frac{(k-q-1)(q-1)n}{n+2-(n-2)q}}$$

We have

$$\int_\Omega u^k(x)dx \leq \frac{\|u\|_{L^{q+1}(\Omega)}^k}{\|h\|_{L^{q+1}(\Omega^*)}^k} \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{\frac{(k-q-1)(1-q)n}{n+2-(n-2)q}} \int_{\Omega^*} h^k(x)dx.$$

If we set

$$C(q,k,\Omega^*) = \int_{\Omega^*} h^k(x)dx \Big/ \|h\|_{L^{q+1}(\Omega^*)}^{\sigma_1},$$

then

$$\int_\Omega u^k(x)dx \leq C(q,k,\Omega^*)\|u\|_{L^{q+1}(\Omega)}^{\sigma_1}$$

and the equality holds if and only Ω is a ball.

If we set

$$C(q,\Omega^*) = \text{ess.}\sup_{x\in\Omega^*} h(x) \Big/ \|h\|^{\sigma_2}_{L^{q+1}(\Omega^*)},$$

then we can obtain

$$\text{ess.}\sup_{x\in\Omega} u(x) \le C(q,\Omega^*)\|u\|^{\sigma_2}_{L^{q+1}(\Omega)}.$$

and the equality holds if and only if Ω is a ball. This completes the proof of Theorem 1.1.

Proof of Corollary 1.2. Following the argument of theorem 1.1, we know that for any $k \ge q+1$,

$$\int_\Omega u^k(x)dx \le \frac{\|u\|^k_{L^{q+1}(\Omega)}}{\|h\|^k_{L^{q+1}(\Omega^*)}}\left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{\frac{(k-q-1)(1-q)n}{n+2-(n-2)q}}\int_{\Omega^*} h^k(x)dx. \tag{4.1}$$

Since $S_q(\Omega) = \|u\|^{q-1}_{L^{q+1}(\Omega)}$ and $S_q(\Omega^*) = \|h\|^{q-1}_{L^{q+1}(\Omega^*)}$, we have

$$\int_\Omega u^k(x)dx \le \frac{S_q^{\frac{k}{q-1}}(\Omega)}{S_q^{\frac{k}{q-1}}(\Omega^*)}\left[\frac{S_q^{\frac{1}{q-1}}(\Omega^*)}{S_q^{\frac{1}{q-1}}(\Omega)}\right]^{\frac{(k-q-1)(1-q)n}{n+2-(n-2)q}}\int_{\Omega^*} h^k(x)dx. \tag{4.2}$$

Noting that $0<q<1$, it follows from Lemma 3.2 that

$$\int_\Omega u^k(x)dx \le \int_{\Omega^*} h^k(x)dx.$$

Consequently

$$\max_{x\in\Omega} u(x) \le \max_{x\in\Omega^*} h(x).$$

Moreover, the equality in the above two inequalities holds if and only if Ω itself is a ball. This completes the proof of Corollary 1.2.

Proof of Corollary 1.3. Let $G(x,y)$ denote the Green's function related to the Laplace operator on Ω^*. Then Green's formula implies that the solution of problem (1.6) can be represented as

$$h(x) = \int_{\Omega^*} G(x,y)h^q(y)dy.$$

Hence

$$\max_{x\in\Omega^*} h(x) \le \max_{x\in\Omega^*} h^q(x) \times \max_{x\in\Omega^*}\int_{\Omega^*} G(x,y)dy \le \left[\max_{x\in\Omega^*} h(x)\right]^q \times \max_{x\in\Omega^*}\int_{\Omega^*} G(x,y)dy.$$

Consequently

$$\max_{x\in\Omega} h(x) \le \left[\max_{x\in\Omega^*}\int_{\Omega^*} G(x,y)dy\right]^{\frac{1}{1-q}}.$$

Let $\gamma(x) = \int_{\Omega^*} G(x,y)dy$. Then it is easy to verify that $\gamma(x)$ satisfies

$$\begin{cases} -\Delta\gamma = 1 & \text{in } \Omega^* \\ \gamma = 0 & \text{on } \partial\Omega^*. \end{cases} \tag{4.3}$$

Hence, an easy computation tells that $\gamma(x) = \left[\frac{|\Omega^*|}{\omega_n(2n)^{\frac{n}{2}}}\right]^{\frac{2}{n}} - \frac{1}{2n}|x|^2$ and $\max\limits_{x\in\Omega^*} \gamma(x) = \left(\frac{|\Omega|}{\omega_n(2n)^{\frac{n}{2}}}\right)^{\frac{2}{n}}$. Thus we have

$$\max_{x\in\Omega} h(x) \leq \left[\max_{x\in\Omega^*} \gamma(x)\right]^{\frac{1}{1-q}} = \left[\frac{|\Omega|}{\omega_n(2n)^{\frac{n}{2}}}\right]^{\frac{2}{(1-q)n}}.$$

Now, the conclusion of Corollary 1.3 follows from Corollary 1.2. This completes the proof of Corollary 1.3.

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